

Spin-1 resonance contributions to the weak Chiral Lagrangian: the vector field formulation *

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Abstract

We use the Vector formulation to evaluate vector and axial-vector exchange contributions to the $\mathcal{O}(p^4)$ weak Chiral Lagrangian. We recover in this framework the bulk of the contributions found previously by Ecker et al. in the antisymmetric formulation of vectors and axial-vectors, but new interesting features arise : i) most of our results are independent of Factorization and ii) novel contributions to non-leptonic kaon decays, proper of this formulation and phenomenologically interesting, are found. The phenomenological implications for $K \rightarrow \pi\pi(\pi)$ and radiative (anomalous and non-anomalous) non-leptonic kaon decays are thus investigated and found particularly relevant.

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1 Introduction

Kaon decays are an important laboratory [1, 2, 3, 4, 5, 6] to understand weak interactions, chiral dynamics and CP violation. Present and future experiments will definitely improve our knowledge in this area [7]. The natural framework to study kaon decays is Chiral Perturbation Theory (χ PT) [8, 9, 10], where matrix elements, consistent with chiral symmetry and its spontaneous breaking ($SU(3)_L \otimes SU(3)_R \rightarrow SU(3)_V$ plus eight Goldstone bosons to be identified with the octet of lightest pseudoscalar mesons), are written as a perturbative expansion in masses and external momenta. At leading $\mathcal{O}(p^2)$ current algebra features like PCAC are recovered.

At next $\mathcal{O}(p^4)$ [8, 9] a loop expansion, required by unitarity, and local operators, consistent with chiral symmetry, appear. The divergent part of the coefficients of these local operators are needed to reabsorb divergences in the loop contributions and thus are called counterterms. The finite part of a counterterm has to be determined from the phenomenology or through extra theoretical assumptions [3] (and references therein).

Vector Meson Dominance (VMD) has proven to be very efficient in describing the coefficients of the $\mathcal{O}(p^4)$ strong lagrangian [11, 12, 13], giving a good agreement with the phenomenology. However VMD in the strong sector is implemented automatically at $\mathcal{O}(p^4)$ only by the antisymmetric formulation of the vectors [14] while in the usual vector formulation has to be just imposed. This is not necessarily the case at higher chiral orders where the conventional vector formulation is able to recover structures required by QCD that the antisymmetric formulation does not generate [15].

We will be interested here in the study of the spin-1 resonance contributions to non-leptonic kaon decays in the χ PT framework. Some of these processes receive contributions already at $\mathcal{O}(p^2)$ (that are not generated by resonance exchange) with sizable $\mathcal{O}(p^4)$ corrections, like $K \rightarrow \pi\pi$ and $K \rightarrow \pi\pi\pi$. Others start at $\mathcal{O}(p^4)$, like $K \rightarrow \pi\gamma^*$, $K \rightarrow \pi\gamma\gamma$, direct emission in $K \rightarrow \pi\pi\gamma$, ... The full weak $\mathcal{O}(p^4)$ counterterm structure has been analysed [16, 17, 18] and there are already interesting counterterm coefficient relations to be considered [1, 6, 19, 20]. The question of the dominance of vector meson exchange in weak decays has not yet a clear answer due to the unknown weak couplings of vector and axial-vector mesons. Thus one has to rely on models to make quantitative predictions. The expansion in inverse power of number of colours ($1/N_c$) has motivated several researches [3, 21] (and references therein) and their consequences for the $\mathcal{O}(p^4)$ weak chiral lagrangian have been studied by direct evaluation of the full short distance weak hamiltonian in the large N_c limit and then bosonizing in terms of the lightest degrees of freedom i.e. the Goldstone boson fields [22]. The general structure of the $\mathcal{O}(p^4)$ weak chiral lagrangian generated by resonance exchange has been studied in Ref. [17] where vectors and axial-vectors are implemented in the antisymmetric formulation and then the Factorization Model (FM), justified by $1/N_c$ arguments, has been used. Several

predictions and useful relations have been found in this way [17].

The questions we address here are:

- i) Do these relations depend on the vector realization? What about if we use the conventional vector formulation instead of the antisymmetric one?
- ii) What is the relative rôle of the Factorization hypothesis in the two formulations?

We will see that the answers provide new results relevant for the phenomenology of most of the processes involved. Moreover we will conclude that the model dependence (factorization) of the results is very much suppressed if the vector formulation of the resonance fields is used.

The scheme of the paper is the following. In Section 2 we review the χ PT formalism in its strong and weak sectors. We also collect and analyse critically the main results of previous work on this topic that we would like to compare with. In Section 3 we deduce the $\mathcal{O}(p^4)$ weak chiral lagrangian as generated by vector and axial-vector contributions in the conventional vector formulation. We compare our results with the ones achieved by the antisymmetric formulation in Ref. [17]. In Section 4 the relevant phenomenological implications of our new results are pointed out. We report our conclusions in Section 5. A brief appendix complements the main text.

2 Chiral Perturbation Theory and weak interactions

In this Section we first review the formulation of χ PT in the treatment of the strong and weak interactions. We also analyse the previous results of the spin-1 resonance exchange generated $\mathcal{O}(p^4)$ weak Chiral Lagrangian using the Factorization Model when the antisymmetric field formulation of the resonance fields is implemented.

2.1 Chiral Perturbation Theory

χ PT [8, 9] is the effective quantum field theory for the study of low energy strong interacting processes. It relies in the exact $G \equiv SU(3)_L \otimes SU(3)_R$ global chiral symmetry of massless QCD. This symmetry group is assumed to break spontaneously to $H \equiv SU(3)_{L+R=V}$ generating an octet of Goldstone bosons that are identified with the lightest octet of pseudoscalar mesons. Following Ref. [23] it turns out convenient to introduce the Goldstone bosons φ_i , $i = 1, \dots, 8$ through the $SU(3)$ matrix $u(\varphi)$, which also parameterizes the coset space G/H as

$$u(\varphi) = \exp \left(\frac{i}{2F} \sum_{j=1}^8 \lambda_j \varphi_j \right) \quad , \quad (1)$$

where λ_i are the $SU(3)$ Gell–Mann matrices ¹ and $F \sim F_\pi \simeq 93 \text{ MeV}$ is the decay constant of the pion. The transformation under $(g_L, g_R) \in G$ is

$$u(\varphi) \xrightarrow{G} g_R u(\varphi) h(g, \varphi)^\dagger = h(g, \varphi) u(\varphi) g_L^\dagger , \quad (2)$$

where the compensator field $h(g, \varphi) \in H$ has been introduced. Green functions and symmetry breaking terms can be more easily generated by promoting the global symmetry to a local one, introducing external fields. A covariant derivative on the $U(\varphi) = uu$ field is then defined as

$$D_\mu U = \partial_\mu U - i r_\mu U + i U \ell_\mu , \quad (3)$$

where $\ell_\mu = v_\mu - a_\mu$ and $r_\mu = v_\mu + a_\mu$, are the left and right external fields, respectively, in terms of the external vector and axial fields.

The leading $\mathcal{O}(p^2)$ strong lagrangian is

$$\mathcal{L}_2 = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle , \quad (4)$$

where

$$\begin{aligned} u_\mu &= i u^\dagger D_\mu U u^\dagger , \\ \chi_\pm &= u^\dagger \chi u^\dagger \pm u \chi^\dagger u , \\ \chi &= 2 B_\circ (s + i p) = 2 B_\circ \mathcal{M} + \dots , \\ \mathcal{M} &= \text{diag}(m_u, m_d, m_s) , \\ B_\circ &= -\frac{1}{F^2} \langle 0 | \bar{u} u | 0 \rangle , \end{aligned} \quad (5)$$

and $\langle A \rangle \equiv \text{Tr}(A)$ in the flavour space. In the definition of χ , s and p are the scalar and pseudoscalar external fields.

Starting at $\mathcal{O}(p^4)$ the strong chiral lagrangian has two well differentiated components corresponding to vertices generating even– and odd–intrinsic parity transitions. The $\mathcal{O}(p^4)$ even–intrinsic parity lagrangian \mathcal{L}_4 was developed in Ref. [9] and can be written as

$$\mathcal{L}_4 = \sum_{i=1}^{10} L_i O_i + H_1 O_{11} + H_2 O_{12} . \quad (6)$$

Here O_i , $i = 1, \dots, 12$ are local operators in terms of the pseudoscalar and external fields. The couplings L_i are rather well determined phenomenologically while H_1 and H_2 are not because the associate operators only involve external fields and therefore do not contribute to low–energy processes with pseudoscalars.

The $\mathcal{O}(p^4)$ odd–intrinsic parity lagrangian arises as a solution to the Ward condition imposed by the chiral anomaly [24].

¹Normalized to $\text{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}$.

The inclusion of other quantum fields than the pseudoscalar Goldstone bosons in the chiral lagrangian was also considered in Ref. [23]. We are interested in the introduction of vector and axial–vector resonances coupled to the non-linear realization of the Goldstone bosons $u(\varphi)$ and to the external fields. It is well known that the incorporation of spin–1 mesons in the chiral lagrangian is not unique and several realizations of the field can be employed [25]. In particular the antisymmetric formulation of vector fields was seen to implement automatically vector meson dominance at $\mathcal{O}(p^4)$ in χ PT [11]. In Ref. [14] was shown that, at $\mathcal{O}(p^4)$ in χ PT, once high energy QCD constraints are taken into account, the usual realizations (antisymmetric, vector, Yang–Mills and Hidden formulations) give equivalent low–energy effective actions. Although the antisymmetric tensor formulation of spin–1 mesons was proven to have a better high–energy behaviour than the vector field realization at $\mathcal{O}(p^4)$ this fact is not necessarily the case at higher orders. In fact for the odd–intrinsic parity action relevant in $V \rightarrow P\gamma$ decays the antisymmetric tensor formulation gives a leading contribution at $\mathcal{O}(p^4)$ while QCD requires explicit $\mathcal{O}(p^3)$ terms that are provided by the vector formulation [15]. The analogous situation in the weak sector has not been studied yet. The authors of Ref. [17] used the antisymmetric fields in their study of the VMD $\mathcal{O}(p^4)$ weak chiral lagrangian. Instead we propose in this work to analyse the conventional vector formulation.

Let us then introduce the nonet of spin-1 resonance fields

$$R_\mu = \frac{1}{\sqrt{2}} \sum_{i=1}^8 \lambda_i R_\mu^i + \frac{1}{\sqrt{3}} R_\mu^0 \quad , \quad R_\mu = V_\mu, A_\mu \quad , \quad (7)$$

that transforms homogeneously under the chiral group as

$$R_\mu \xrightarrow{G} h(g, \varphi) R_\mu h(g, \varphi)^\dagger \quad . \quad (8)$$

The mixing between the eighth and the singlet components of the vector field is assumed ideal, i.e. $V_8^\mu = (\omega^\mu + \sqrt{2} \phi^\mu) / \sqrt{3}$, consistently with the phenomenology. There is no such a statement in the axial–vector case but nevertheless we will do the same assumption here (in any case deviations from this would affect very marginally our results).

The kinetic term for the resonance field is

$$\mathcal{L}_K = -\frac{1}{4} \langle R_{\mu\nu} R^{\mu\nu} \rangle + \frac{m_R^2}{2} \langle R_\mu R^\mu \rangle \quad , \quad (9)$$

where $R_{\mu\nu} = \nabla_\mu R_\nu - \nabla_\nu R_\mu$ and ∇_μ is the covariant derivative defined in Ref. [11] as

$$\nabla_\mu A = \partial_\mu A + [\Gamma_\mu, A] \quad , \quad (10)$$

$$\Gamma_\mu \equiv \frac{1}{2} \left\{ u^\dagger (\partial_\mu - i r_\mu) u + u (\partial_\mu - i \ell_\mu) u^\dagger \right\} \quad ,$$

for any A operator that transforms homogeneously as the resonant field in Eq. (8).

The most general strong lagrangian linear in the vector field and up to $\mathcal{O}(p^3)$, assuming nonet symmetry, reads [14, 26]

$$\begin{aligned}\mathcal{L}_V = & -\frac{f_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu} \rangle - \frac{ig_V}{2\sqrt{2}}\langle V_{\mu\nu} [u^\mu, u^\nu] \rangle \\ & + i\alpha_V \langle V_\mu [u_\nu, f_-^{\mu\nu}] \rangle + \beta_V \langle V_\mu [u^\mu, \chi_-] \rangle \\ & + h_V \varepsilon_{\mu\nu\rho\sigma} \langle V^\mu \{u^\nu, f_+^{\rho\sigma}\} \rangle + i\theta_V \varepsilon_{\mu\nu\rho\sigma} \langle V^\mu u^\nu u^\rho u^\sigma \rangle ,\end{aligned}\quad (11)$$

where

$$f_\pm^{\mu\nu} = u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u , \quad (12)$$

and $F_{R,L}^{\mu\nu}$ are the strength field tensors associated to the external r_μ and ℓ_μ fields, respectively. The couplings in \mathcal{L}_V can be determined, in principle, from the phenomenology of the vector meson decays. Thus $|f_V|$, $|h_V|$, $|\theta_V|$ and $|\alpha_V|$ could be obtained from the experimental widths [27] of $\rho^0 \rightarrow e^+e^-$, $\omega \rightarrow \pi^0\gamma$, $\omega \rightarrow \pi\pi\pi$ and $\rho \rightarrow \pi\pi\gamma$, respectively, while g_V and β_V enter in $\rho \rightarrow \pi\pi$. In Table 1 we collect the experimental determinations (when available) and also the predictions of the Hidden Symmetry model (HS) [13]² and Extended Nambu–Jona–Lasinio model (ENJL) [26]³. The relative signs of the couplings are well fixed from the phenomenology. The positive slope of the $\pi^0 \rightarrow \gamma\gamma^*$ form factor implies that $f_V h_V > 0$, while the charge radius of the pion together with the assumption of VMD gives $f_V g_V > 0$. Finally we get $g_V \beta_V < 0$ from a combined analysis of $\Gamma(\rho \rightarrow \pi\pi)$ and $\Gamma(\phi \rightarrow K^+K^-)$. We see that the model predictions agree with these results. In any case we are more interested in the general consequences of vector meson exchange than in an accurate numerical estimate, unrealistic with the present knowledge.

Analogously the most general strong lagrangian linear in the axial–vector field and up to $\mathcal{O}(p^3)$, assuming nonet symmetry, reads [14, 26]

$$\begin{aligned}\mathcal{L}_A = & -\frac{f_A}{2\sqrt{2}}\langle A_{\mu\nu} f_-^{\mu\nu} \rangle + i\alpha_A \langle A_\mu [u_\nu, f_+^{\mu\nu}] \rangle + \gamma_1 \langle A_\mu u_\nu u^\mu u^\nu \rangle \\ & + \gamma_2 \langle A_\mu \{u^\mu, u^\nu u_\nu\} \rangle + \gamma_3 \langle A_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle + \gamma_4 \langle A_\mu u^\mu \rangle \langle u_\nu u^\nu \rangle \\ & + h_A \varepsilon_{\mu\nu\rho\sigma} \langle A^\mu \{u^\nu, f_-^{\rho\sigma}\} \rangle .\end{aligned}\quad (13)$$

The couplings of \mathcal{L}_A can be obtained from the study of the axial–vector decays. Thus f_A and α_A enter in $a_1 \rightarrow \pi\gamma$, h_A in $a_1 \rightarrow \pi\pi\gamma$ and γ_i , $i = 1, \dots, 4$ in $a_1 \rightarrow \pi(\pi\pi)_{S\text{-wave}}$ and $f_1(1285) \rightarrow \eta\pi\pi$, $K\bar{K}\pi$, $\pi\pi\pi\pi$. However the experimental situation is still poor [27] and the only rather well known coupling is f_A [29]. Therefore we will rely, for the rest, in the predictions of the ENJL model [26]. We collect the $\mathcal{O}(p^3)$ axial–vector couplings in Table 2.

We would like to emphasize the fact that the strong sector in the conventional vector formulation, at leading $\mathcal{O}(p^3)$, is much richer than the leading $\mathcal{O}(p^2)$ strong action in the

²In Ref. [28] we evaluated h_V in the HS model.

³We thank F.J. Botella for pointing out to us a few mistakes in the evaluation of the couplings in Ref. [26] and for providing us with his own evaluation of the couplings in the ENJL model.

	HS	ENJL	Expt.
f_V	input	0.17	0.20
g_V	$f_V/2$	0.08	0.09
h_V	0.039	0.033	0.037
θ_V	$2h_V$	0.050	-
α_V	-	-0.015	-
β_V	-	-0.015	-0.018

Table 1: $\mathcal{O}(p^3)$ vector couplings in the Hidden Symmetry model (HS), Extended Nambu–Jona–Lasinio model (ENJL) and the experiment (Expt.) when available. We do not quote the experimental errors (typically $\simeq 10\%$) since higher chiral order corrections, difficult to estimate and in principle larger, have not been considered yet.

	ENJL	Expt.
f_A	0.085	0.097
h_A	0.014	-
α_A	-0.009	-
γ_1	0.004	-
γ_2	-0.010	-
$\gamma_{3,4}$	$\mathcal{O}(1/\sqrt{N_c})$	-

Table 2: $\mathcal{O}(p^3)$ axial–vector couplings in the Extended Nambu–Jona–Lasinio model (ENJL) and the experimental value (Expt.) when available. About the experimental error of f_A see the explanation in the caption of Table 1.

antisymmetric fields [11]. Terms analogous to $\alpha_{V,A}$, β_V , γ_i , $h_{V,A}$ and θ_V are not present in this last formulation at leading order.

For a further extensive and thorough exposition on χ PT see Refs. [4, 5].

2.2 Non-leptonic weak interactions in χ PT

The complete non-leptonic $\Delta S = 1$ effective weak hamiltonian at low energies ($E \ll M_W$) for the lightest degrees of freedom (u,d,s) is constructed through an OPE expansion. It is given by a sum of products of Wilson coefficients $C_i(\mu)$ and four-quark local operators $Q_i(\mu)$ $i = 1, \dots, 6$, [30, 31, 32] and reads

$$\mathcal{H}_{NL}^{|\Delta S|=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^6 C_i(\mu) Q_i + h.c. \quad (14)$$

Here G_F is the Fermi constant and V_{ij} are elements of the CKM-matrix. These operators (or combinations of them) belong to the $(8_L, 1_R)$ and $(27_L, 1_R)$ representations of the chiral group G. The first ones give only $\Delta I = 1/2$ transitions while the seconds generate both $\Delta I = 1/2, 3/2$ transitions. In the following we will neglect the contributions coming from $(27_L, 1_R)$ operators.

At $\mathcal{O}(p^2)$ in χ PT, we can construct only one relevant $|\Delta S| = 1$ operator which transforms as an octet under the chiral group

$$\begin{aligned} \mathcal{L}_2^{|\Delta S|=1} &= 4 G_8 \langle \lambda_6 L_1^\mu L_\mu^1 \rangle \\ &= G_8 F^4 \langle \Delta u_\mu u^\mu \rangle, \end{aligned} \quad (15)$$

where

$$L_\mu^1 = \frac{\delta S_2^x}{\delta \ell^\mu} = -i \frac{F^2}{2} U^\dagger D_\mu U = -\frac{F^2}{2} u^\dagger u_\mu u, \quad (16)$$

is the left-handed current associated to the $\mathcal{O}(p^2)$ strong lagrangian in Eq. (4) and $\Delta = u \lambda_6 u^\dagger$. From the experimental width of $K \rightarrow \pi\pi$ one gets ⁴

$$|G_8|_{K \rightarrow \pi\pi} \equiv |G_8| \simeq 9.2 \times 10^{-6} \text{ GeV}^{-2}. \quad (17)$$

At $\mathcal{O}(p^4)$ the chiral weak lagrangian has been studied in Refs. [16, 17, 18] giving 37 chiral operators W_i only in the octet part

$$\mathcal{L}_4^{|\Delta S|=1} = G_8 F^2 \sum_{i=1}^{37} N_i W_i, \quad (18)$$

that introduce 37 new unknown coupling constants N_i . Their phenomenological determination is then very difficult. Nevertheless the theory is already predictive at this level due

⁴If $\mathcal{O}(p^4)$ corrections are taken into account, a phenomenological value of $|G_8| \simeq 6.5 \times 10^{-6} \text{ GeV}^{-2}$ is obtained [33].

to the fact that the symmetry provides relations between processes. Thus, for instance, the same $\mathcal{O}(p^4)$ weak counterterm combination appears in the structure dependent electric amplitudes to $K_S \rightarrow \pi^+\pi^-\gamma$, $K^+ \rightarrow \pi^+\pi^0\gamma$ and $K_L(K^\pm) \rightarrow \pi\pi\pi\gamma$ [19, 34, 35, 36]. Unfortunately the actual status of the phenomenology is not good enough to provide absolute quantitative predictions and to do so one has to rely on models.

Our purpose in this paper is to study the spin-1 resonance exchange contributions to the N_i couplings and to clarify how much can be said in a model-independent way.

2.3 The Factorization Model and the Spin-1 resonance exchange contributions in the antisymmetric formulation

The Factorization Model (FM) has been used in the context of vector and axial-vector meson dominance [17]. If we neglect the penguin contributions, justified by the $1/N_c$ expansion [21], we can write the octet dominant piece in Eq. (14) as

$$\mathcal{H}_{NL}^{|\Delta S|=1} = -\frac{G_F}{2\sqrt{2}} V_{ud}V_{us}^* C_-(\mu) Q_- + h.c. , \quad (19)$$

with

$$Q_- = 4(\bar{s}_L \gamma^\mu u_L)(\bar{u}_L \gamma_\mu d_L) - 4(\bar{s}_L \gamma^\mu d_L)(\bar{u}_L \gamma_\mu u_L) , \quad (20)$$

and $(\bar{s}_L \gamma_\mu u_L) \equiv \frac{1}{2} \bar{s}^\alpha \gamma_\mu (1 - \gamma_5) u_\alpha$ (α is a colour index). In a chiral gauge theory the quark bilinears in the Q_- operator are given by the associated left-handed current

$$\frac{\delta S}{\delta \ell^\mu} \equiv L_\mu = L_\mu^1 + L_\mu^3 + L_\mu^5 + \dots, \quad (21)$$

(the first term L_μ^1 was already given in Eq. (16)) where $S[U, \ell, r, s, p]$ is the low-energy strong effective action of QCD in terms of the Goldstone bosons realization U and the external fields ℓ, r, s, p .

Factorization, in the context of χ PT, amounts to the assumption that the product of quark bilinears in Q_- factorizes in the *current* \times *current* form [21] as

$$\mathcal{L}_{FM} = 4 k_F G_8 \langle \lambda \frac{\delta S}{\delta \ell_\mu} \frac{\delta S}{\delta \ell^\mu} \rangle + h.c. , \quad (22)$$

where $\lambda \equiv \frac{1}{2}(\lambda_6 - i\lambda_7)$ and G_8 has been defined in Eqs. (15,17). The FM in (22) gives a full prediction but for an overall fudge factor k_F that is not given by the model. In general $k_F \simeq \mathcal{O}(1)$ and naive factorization would imply $k_F \simeq 1$. Hence the long time standing problem of the $\Delta I = 1/2$ enhancement can be formulated like this: while the perturbative evaluation of $C_-(\mu)$ in Eq. (19) implies a small value of k_F [31, 32]

$$C_-(m_\rho) \simeq 2.2 \quad \longrightarrow \quad k_F \simeq 0.2 - 0.3 , \quad (23)$$

the phenomenology of the $K \rightarrow \pi\pi(\pi)$ processes requires $k_F \simeq 1$. In this case is rather clear that the FM only provides a parameterization without any physical insight in the

problem, i.e. parameterizes uncontrolled non-factorizable contributions. However if in other processes one finds instead that $k_F \simeq 0.2 - 0.3$ (as predicted by the Wilson coefficient) then factorization has a clear dynamical meaning.

The rôle of factorization is also important in uncovering the physical significance of the chiral structures of the weak lagrangian. For instance the operators W_{28}, \dots, W_{31} in $\mathcal{L}_4^{|\Delta S|=1}$ have been proven to have an interesting dynamical origin : they can be generated by the chiral anomaly [37, 38] as a factorizable contribution in which one of the currents comes from the Wess–Zumino–Witten action [24]. Moreover in this particular case non-factorizable contributions cannot give different chiral structures to the operators.

In Ref. [17] the resonance exchange contributions to $\mathcal{L}_4^{|\Delta S|=1}$ have been studied using the antisymmetric formulation for the spin-1 fields. After a model-independent analysis that specifies the N_i couplings in terms of a rather large set of unknown weak couplings involving resonances (that is already able to provide a list of relations between the couplings) and in order to increase the predictive power, the authors of that reference use the FM to be able to specify the N_i couplings in terms of only one parameter : the k_F factorization factor in Eq. (22).

As already pointed out in Ref. [17], there are two ways to derive a FM weak lagrangian generated by resonance exchange (we focus here on the spin-1 fields) :

- (\mathcal{A}) To evaluate the strong action generated by resonance exchange, and then perform the factorization procedure in Eq. (22). Since we apply the FM procedure once the vectors have already been integrated out, we can say that we have performed factorization at the scale of the low energy effective field theory i.e. the Goldstone boson mass scale. Consequently the lagrangian has been generated at the kaon mass scale.
- (\mathcal{B}) Conversely we can write down the spin-1 strong chiral lagrangian, we apply factorization and thus we derive the weak resonance couplings. Then we integrate out the resonance fields and generate the effective lagrangian that, in this way, is generated at the scale of the resonance.

In principle the two effective actions, thus generated, do not have to coincide.

Actually we have shown in a previous work [28] that the weak $\mathcal{O}(p^6)$ VMD lagrangian for $K_L \rightarrow \gamma\gamma^*$ and $K_L \rightarrow \pi^0\gamma\gamma$ obtained with procedure \mathcal{B} has extra chiral structures compared to the ones obtained with procedure \mathcal{A} . Both give a good phenomenological description, but with different factorization factor: $k_F \simeq 1$ for method \mathcal{A} , while $k_F \simeq 0.2 - 0.3$ (i.e. the Wilson coefficient) for the scheme \mathcal{B} . Also the weak $\mathcal{O}(p^6)$ VMD lagrangian for $K_L \rightarrow \pi^+\pi^-\gamma$ shares this property: procedure \mathcal{B} has a more complete set of operators [39] compared to the one obtained with procedure \mathcal{A} [38]. Method \mathcal{B} is what we define as factorization in the vectors: FMV model [28].

To generate $\mathcal{L}_4^{|\Delta S|=1}$ in Eq. (18) procedure \mathcal{A} requires the determination of the VMD strong $\mathcal{O}(p^4)$ action, i.e. the VMD contribution to the \mathcal{L}_4 in Eq. (6). This VMD action has already been determined in the antisymmetric formulation [11]. If resonance saturation of \mathcal{L}_4 is assumed, both procedures \mathcal{A} and \mathcal{B} generate the same $\mathcal{L}_4^{|\Delta S|=1}$ effective action in this formulation [17]. However a word of caution is necessary in relation with the H_1 coupling in \mathcal{L}_4 . This coupling also receives a resonant contribution [11] but, as already commented before, is not measurable. The authors of Ref. [17] suggested to leave free this coupling (as we will comment below, in their phenomenological study of $K \rightarrow \pi\gamma^*$ they input $H_1 = 0$ at the scale $\mu = m_\rho$). This amounts to add local contributions into the lagrangian (therefore spoiling the assumption of resonance saturation) and then \mathcal{A} and \mathcal{B} schemes differ in this case.

It should also be mentioned that in Ref. [17] the scalar and pseudoscalar resonance exchange were also studied. When potentially relevant ($K \rightarrow \pi\pi(\pi)$), we will comment on it.

3 $\mathcal{O}(p^4)$ weak counterterms from spin-1 resonances in the vector formulation

The $\mathcal{O}(p^3)$ strong lagrangian of vectors and axial-vectors has been given in Eqs. (11,13) respectively. Thus to describe vector and axial-vector exchange contributions at $\mathcal{O}(p^4)$ in the weak chiral lagrangian $\mathcal{L}_4^{|\Delta S|=1}$ in Eq. (18) we need an $\mathcal{O}(p)$ CP conserving effective weak lagrangian linear in the vector and axial-vector fields.

Indeed, assuming octet dominance, there are only two such operators transforming as $(8_L, 1_R)$ under the chiral group :

$$\mathcal{L}_R^{\mathcal{O}(p)} = G_8 F_\pi^4 \left[\omega_1^R \langle \Delta \{ R_\mu, u^\mu \} \rangle + \omega_2^R \langle \Delta u_\mu \rangle \langle R^\mu \rangle \right] , \quad (24)$$

where $R_\mu = V_\mu, A_\mu$, and Δ has been defined in connection with Eq. (15). The operators in Eq. (24) generate $\mathcal{O}(p)$ weak decays of vectors (axials) into two or more pseudoscalars with unknown couplings ω_1^R and ω_2^R . To evaluate the vector (axial) contribution to $\mathcal{O}(p^4)$ weak lagrangian $\mathcal{L}_4^{|\Delta S|=1}$ (18) we can just integrate out the resonance fields between $\mathcal{L}_R^{\mathcal{O}(p)}$ and \mathcal{L}_V and \mathcal{L}_A in Eqs. (11,13) respectively. Equivalently we can derive the $\mathcal{O}(p^4)$ weak lagrangian just redefining the vector (axial) field as

$$R_\mu \longrightarrow R_\mu - \frac{G_8 F_\pi^4}{m_R^2} \left[\omega_1^R \{ \Delta, u_\mu \} + \omega_2^R \langle \Delta u_\mu \rangle \right] , \quad (25)$$

in \mathcal{L}_V and \mathcal{L}_A .

The relevant features of this procedure are the following :

- i) The weak field redefinitions eliminate the weak $\mathcal{O}(p)$ resonance couplings in $\mathcal{L}_R^{\mathcal{O}(p)}$ Eq. (24) generating vector and axial–vector exchange contributions to the $\mathcal{O}(p^4)$ weak lagrangian $\mathcal{L}_4^{|\Delta S|=1}$.
- ii) The kinetic term of the resonances in Eq. (9) generates also an $\mathcal{O}(p^3)$ weak lagrangian for the vectors (axials) irrelevant at $\mathcal{O}(p^4)$, but necessary to derive in a complete and consistent manner the weak $\mathcal{O}(p^6)$ VMD lagrangian, as we will comment at the end of this Section.

In this formulation the non-vanishing contributions to $\mathcal{L}_4^{|\Delta S|=1}$ in Eq. (18) satisfy the following relations:

1/ Vector contributions.

$$\begin{aligned}
N_1^V &= -N_2^V = -N_{19}^V = N_{16}^V + \frac{N_{17}^V}{2}, \\
2N_{14}^V &= N_{15}^V = -4N_{18}^V = N_{25}^V = 4N_{27}^V = -4N_{37}^V, \\
N_{29}^V &= N_{34}^V, \\
g_V N_{17}^V &= -2\sqrt{2}\alpha_V N_1^V, \\
f_V N_{29}^V &= \frac{h_V}{\sqrt{2}} N_{15}^V, \\
f_V N_1^V &= -g_V N_{15}^V, \\
2h_V N_{28}^V &= \theta_V N_{30}^V, \\
g_V N_9^V &= -\sqrt{2}\beta_V N_1^V.
\end{aligned} \tag{26}$$

2/ Axial–vector contributions.

$$\begin{aligned}
N_{15}^A &= -2N_{14}^A, \\
2N_{16}^A &= N_{17}^A = \frac{4}{3}N_{18}^A = N_{26}^A = -4N_{27}^A = -4N_{37}^A, \\
N_{29}^A &= -N_{34}^A, \\
2\sqrt{2}\alpha_A N_{17}^A &= f_A N_{15}^A, \\
h_A N_{17}^A &= -\sqrt{2}f_A N_{29}^A.
\end{aligned} \tag{27}$$

Comparing these relations with the ones obtained by Ecker et al. in the antisymmetric formulation [17] (see their Eqs. (3.18,3.19) for the vectors and Eq. (3.20) for the axials) we see that most of their relations are satisfied in our framework using $f_V = 2g_V$ (as they do). The only exceptions correspond to our results in the combinations $N_1^V - 2N_2^V + 3N_{14}^V + 3N_{16}^V - 6N_{18}^V = -3F_\pi^2 \alpha_V \omega_1^V/m_V^2$ and $N_{16}^V - N_{18}^V + N_{27}^V = -F_\pi^2 \alpha_V \omega_1^V/m_V^2$ both of which they find to vanish. In any case our relations in Eqs. (26,27) are many more (some of them independent of the assumptions $f_V = 2g_V$ and $\theta_V = 2h_V$). This is due

to the fact that while we have a richer structure on the strong sector (as emphasized before) our leading weak $\mathcal{O}(p)$ lagrangian $\mathcal{L}_R^{\mathcal{O}(p)}$ in Eq. (24) is much more restricted in comparison with the leading $\mathcal{O}(p^2)$ weak lagrangian for spin-1 resonance contribution in the antisymmetric formalism [17]. Both realizations get therefore a balance. The vector formalism is richer in the strong sector while the antisymmetric tensor fields give a richer structure to the weak one. The fact that the main rôle is given to the rather well known strong couplings in the conventional vector fields offers a much more predictive scheme in a model-independent way.

In the third and fourth columns of Tables 3,4 and 5 we report the spin-1 resonance contributions to the couplings N_i of the phenomenologically relevant W_i operators. For the ones that are not involved in the relevant decays : N_{19} , N_{25} , N_{26} , N_{27} , N_{34} and N_{37} we refer to the Eqs. (26,27) above (the structures of the corresponding W_i operators can be read from Ref. [17]). The remaining operators in $\mathcal{L}_4^{|\Delta S|=1}$ do not get any contributions.

In our framework the rôle of the model dependence (factorization) has a very marginal impact : one finds only a relation among ω_1^R and ω_2^R ($R = V, A$) in $\mathcal{L}_R^{\mathcal{O}(p)}$ in Eq. (24). Indeed we find (see Appendix A) :

$$\omega_1^R = \sqrt{2} \frac{m_R^2}{F_\pi^2} f_R \eta_R = -\omega_2^R, \quad R = V, A, \quad (28)$$

where η_R , $R = V, A$ is the $\mathcal{O}(1)$ unknown factorization factor (in principle $\eta_V \neq \eta_A$). If the $\Delta I = 1/2$ enhancement is at work then $\eta_R \simeq 1$, while $\eta_R \simeq 0.2 - 0.3$ if it is given by the Wilson coefficient. In Eq. (28) f_R , $R = V, A$ are the strong couplings defined in \mathcal{L}_V (Eq. (11)) and \mathcal{L}_A (Eq. (13)).

In the fifth and sixth column of Tables 3,4 and 5 we rewrite the spin-1 resonance contribution to N_i using Eq. (28) for the relevant phenomenological operators.

Notice that in those couplings where only one of the two ω_i^R appears (independently for $R = V, A$), i.e. all the relatives to non-radiative non-leptonic kaon decays (Table 3) (but for the axial contribution to N_4), and all the relatives to the non-anomalous radiative non-leptonic kaon decays (Table 4) the use of factorization (Eq. (28)) just amounts to rewriting the couplings without extra information : they are model-independent.

The assumption of factorization only enters in the operators in Table 5 that contribute to anomalous radiative non-leptonic kaon decays (N_{28}^V and N_{30}^V depend on ω_2^V and N_{31}^A depend on ω_2^A). In fact we get two extra relations :

$$\begin{aligned} N_{30}^V &= 4 N_{29}^V, \\ N_{31}^A &= -4 N_{29}^A. \end{aligned} \quad (29)$$

Also by comparison to Ref. [17] we do get vector and axial-vector contributions to the anomalous N_{28}, \dots, N_{31} , while factorization in the antisymmetric formulation gives a vanishing contribution. In the remaining vector contributions we agree completely, while

i	W_i	Vectors	Axial-Vectors	Expressions using $\omega_1^R = \sqrt{2} \frac{m_R^2}{F_\pi^2} f_R \eta_R, \quad \omega_2^R = -\omega_1^R$	
				Vectors	Axial-Vectors
1	$\langle \Delta u_\mu u^\mu u_\nu u^\nu \rangle$	$-\frac{F_\pi^2}{m_V^2} \frac{g_V}{\sqrt{2}} \omega_1^V$	$\frac{2}{3} \frac{F_\pi^2}{m_A^2} (\gamma_1 - 3\gamma_2 - 4\gamma_4) \omega_1^A$	$-f_V g_V \eta_V$	$\frac{2\sqrt{2}}{3} f_A (\gamma_1 - 3\gamma_2 - 4\gamma_4) \eta_A$
2	$\langle \Delta u_\mu u_\nu u^\nu u^\mu \rangle$	$\frac{F_\pi^2}{m_V^2} \frac{g_V}{\sqrt{2}} \omega_1^V$	$\frac{2}{3} \frac{F_\pi^2}{m_A^2} (2\gamma_1 - 3\gamma_2 - 2\gamma_4) \omega_1^A$	$f_V g_V \eta_V$	$\frac{2\sqrt{2}}{3} f_A (2\gamma_1 - 3\gamma_2 - 2\gamma_4) \eta_A$
3	$\langle \Delta u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle$	-	$-2 \frac{F_\pi^2}{m_A^2} (\gamma_1 + \gamma_3) \omega_1^A$	-	$-2\sqrt{2} f_A (\gamma_1 + \gamma_3) \eta_A$
4	$\langle \Delta u_\mu \rangle \langle u^\mu u_\nu u^\nu \rangle$	-	$-\frac{F_\pi^2}{m_A^2} \left[\frac{4}{3} (\gamma_1 - \gamma_4) \omega_1^A + (\gamma_1 + 2\gamma_2) \omega_2^A \right]$	-	$-\frac{\sqrt{2}}{3} f_A (\gamma_1 - 6\gamma_2 - 4\gamma_4) \eta_A$
9	$\langle \Delta [\chi_-, u_\mu u^\mu] \rangle$	$\frac{F_\pi^2}{m_V^2} \beta_V \omega_1^V$	-	$\sqrt{2} f_V \beta_V \eta_V$	-

Table 3: Vector and Axial-vector contribution to the N_i coefficients of the W_i octet operators, in the basis of Ref. [17], relevant to pure non-leptonic kaon decays at $O(G_F)$. The hypothesis of factorization is only used to relate ω_1^A with ω_2^A in the operator W_4 .

i	W_i	Vectors	Axial-Vectors	Expressions using $\omega_1^R = \sqrt{2} \frac{m_R^2}{F_\pi^2} f_R \eta_R$	
				Vectors	Axial-Vectors
14	$i\langle\Delta\{f_+^{\mu\nu}, u_\mu u_\nu\}\rangle$	$\frac{F_\pi^2}{m_V^2} \frac{f_V}{2\sqrt{2}} \omega_1^V$	$-\frac{F_\pi^2}{m_A^2} \alpha_A \omega_1^A$	$\frac{1}{2} f_V^2 \eta_V$	$-\sqrt{2} f_A \alpha_A \eta_A$
15	$i\langle\Delta u_\mu f_+^{\mu\nu} u_\nu\rangle$	$\frac{F_\pi^2}{m_V^2} \frac{f_V}{\sqrt{2}} \omega_1^V$	$2 \frac{F_\pi^2}{m_A^2} \alpha_A \omega_1^A$	$f_V^2 \eta_V$	$2\sqrt{2} f_A \alpha_A \eta_A$
16	$i\langle\Delta\{f_-^{\mu\nu}, u_\mu u_\nu\}\rangle$	$-\frac{F_\pi^2}{m_V^2} \left(\frac{g_V}{\sqrt{2}} + \alpha_V\right) \omega_1^V$	$\frac{F_\pi^2}{m_A^2} \frac{f_A}{2\sqrt{2}} \omega_1^A$	$-f_V(g_V + \sqrt{2}\alpha_V)\eta_V$	$\frac{1}{2} f_A^2 \eta_A$
17	$i\langle\Delta u_\mu f_-^{\mu\nu} u_\nu\rangle$	$2 \frac{F_\pi^2}{m_V^2} \alpha_V \omega_1^V$	$\frac{F_\pi^2}{m_A^2} \frac{f_A}{\sqrt{2}} \omega_1^A$	$2\sqrt{2} f_V \alpha_V \eta_V$	$f_A^2 \eta_A$
18	$\langle\Delta(f_{+\mu\nu}^2 - f_{-\mu\nu}^2)\rangle$	$-\frac{F_\pi^2}{m_V^2} \frac{f_V}{4\sqrt{2}} \omega_1^V$	$\frac{3}{4\sqrt{2}} \frac{F_\pi^2}{m_A^2} f_A \omega_1^A$	$-\frac{1}{4} f_V^2 \eta_V$	$\frac{3}{4} f_A^2 \eta_A$

Table 4: Vector and Axial-vector contribution to the N_i coefficients of the W_i octet operators, in the basis of Ref.[17], relevant to radiative non-anomalous non-leptonic kaon decays at $O(G_F)$. Notice that the factorization hypothesis is not used anywhere in this Table.

i	W_i	Vectors	Axial-Vectors	Expressions using $\omega_1^R = \sqrt{2} \frac{m_B^2}{F_\pi^2} f_R \eta_R, \quad \omega_2^R = -\omega_1^R$	
				Vectors	Axial-Vectors
28	$i\varepsilon_{\mu\nu\rho\sigma} \langle \Delta u^\mu \rangle \langle u^\nu u^\rho u^\sigma \rangle$	$-\frac{F_\pi^2}{m_V^2} \theta_V \omega_2^V$	-	$\sqrt{2} f_V \theta_V \eta_V$	-
29	$\varepsilon_{\mu\nu\rho\sigma} \langle \Delta [f_+^{\rho\sigma} - f_-^{\rho\sigma}, u^\mu u^\nu] \rangle$	$\frac{F_\pi^2}{m_V^2} \frac{h_V}{2} \omega_1^V$	$-\frac{F_\pi^2}{m_A^2} \frac{h_A}{2} \omega_1^A$	$\frac{1}{\sqrt{2}} f_V h_V \eta_V$	$-\frac{1}{\sqrt{2}} f_A h_A \eta_A$
30	$\varepsilon_{\mu\nu\rho\sigma} \langle \Delta u^\mu \rangle \langle f_+^{\rho\sigma} u^\nu \rangle$	$-2 \frac{F_\pi^2}{m_V^2} h_V \omega_2^V$	-	$2\sqrt{2} f_V h_V \eta_V$	-
31	$\varepsilon_{\mu\nu\rho\sigma} \langle \Delta u^\mu \rangle \langle f_-^{\rho\sigma} u^\nu \rangle$	-	$-2 \frac{F_\pi^2}{m_A^2} h_A \omega_2^A$	-	$2\sqrt{2} f_A h_A \eta_A$

Table 5: Vector and Axial-vectors contribution to the N_i coefficients of the W_i octet operators, in the basis of Ref.[17], relevant to radiative anomalous non-leptonic kaon decays at $O(G_F)$. The hypothesis of factorization is only used to relate ω_1^R with ω_2^R .

for the axial-vectors we disagree with Ref. [17] only in N_{16}^A and N_{17}^A : N_{17}^A is vanishing and N_{16}^A is twice our value in their formulation. We stress also that for the other axial contributions like N_{18}^A we agree, and for us the relative weights among all axial couplings, (see Eq. (27)) are independent of factorization. Thus we attribute this difference to the diverse formulation; of course this has physical consequences, as we will comment in the discussion on the phenomenology.

Regarding the two different procedures: (\mathcal{A}) to integrate the vectors first and then perform FM, or (\mathcal{B}) vice versa (see discussion at the end of Section 2.3) here, so far we have used only procedure \mathcal{B} . If we assume that the $\mathcal{O}(p^4)$ VMD strong lagrangian is the same as the antisymmetric formulation [14] procedures \mathcal{B} and \mathcal{A} differ just as much as we differ from the FM results of Ref. [17].

As already commented at the end of Section 2.3, in the weak $\mathcal{O}(p^6)$ VMD lagrangian for $K_L \rightarrow \gamma\gamma^*$ and $K_L \rightarrow \pi^0\gamma\gamma$ procedure \mathcal{B} produces all the structures generated by procedure \mathcal{A} , plus additional ones. In fact the $\mathcal{O}(p^3)$ vector weak lagrangian induced by the shift in Eq. (25) in the kinetic term is needed in order to recover completely in procedure \mathcal{B} the structure generated by procedure \mathcal{A} [38], as we have shown explicitly in the weak $\mathcal{O}(p^6)$ VMD lagrangian for $K_L \rightarrow \pi^+\pi^-\gamma$ [39].

This gives, we think, also more reliability to our model which describes simultaneously, in a consistent and complete manner, the weak $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ VMD lagrangians in the conventional vector formulation.

4 Discussion on the Phenomenology

The new features of the framework we have proposed in the last Section provide important consequences on the phenomenology of many non-leptonic kaon decays. We will discuss them here in turn.

4.1 $K \rightarrow \pi\pi(\pi)$

$K \rightarrow 3\pi$ amplitudes, due to the small phase space available, are generally expanded in the kinetic energy of the pions up to quadratic slopes and decomposed, jointly with $K \rightarrow \pi\pi$, according to the isospin of the final state [6, 33, 40]. Then the isospin amplitudes are determined by fitting the full set of data [33]. If we neglect the $\Delta I = 3/2$ transitions, the determined amplitudes are A_0 for $K \rightarrow \pi\pi$ and in $K \rightarrow 3\pi$ the amplitude at the center of the Dalitz plot (α_1), the linear slope (β_1) and the quadratic slopes (ζ_1 and ξ_1).

The chiral expansion proves to be predictive. $K \rightarrow \pi\pi(\pi)$ start at $\mathcal{O}(p^2)$ in χ PT and receive sizable $\mathcal{O}(p^4)$ loop and counterterm contributions [33]. The counterterms have two possible sources:

- i) Weak transitions in the external legs of pole diagrams that are determined by the strong L_i couplings in \mathcal{L}_4 .
- ii) Direct weak terms that are provided by the weak N_i couplings in $\mathcal{L}_4^{|\Delta S|=1}$.

The experimentally determined $\Delta I = 1/2$ amplitudes and slopes receive the following weak $\mathcal{O}(p^4)$ counterterm contributions [33] :

$$\begin{aligned}
A_0 &= -\frac{m_K^2}{F_\pi^2 F_K} \frac{2}{3} \sqrt{\frac{2}{3}} (m_K^2 - m_\pi^2) K_1 , \\
\alpha_1 &= -\frac{2 m_K^4}{27 F_K F_\pi^3} \left[(K_1 - K_2) + 24 G_8 F_\pi^2 (2L_1 + 2L_2 + L_3) \right] , \\
\beta_1 &= -\frac{m_K^2 m_\pi^2}{9 F_K F_\pi^3} \left[(K_3 - 2K_1) + 24 G_8 F_\pi^2 (-2L_1 + L_2 - L_3 + 12L_4) \right] , \\
\zeta_1 &= -\frac{m_\pi^4}{6 F_K F_\pi^3} \left[K_2 - 24 G_8 F_\pi^2 (2L_1 + 2L_2 + L_3) \right] , \\
\xi_1 &= -\frac{m_\pi^4}{6 F_K F_\pi^3} \left[K_3 - 24 G_8 F_\pi^2 (2L_1 - L_2 + L_3) \right] .
\end{aligned} \tag{30}$$

Here K_1 , K_2 and K_3 are three scale dependent weak $\mathcal{O}(p^4)$ counterterm combinations in terms of N_i . In the second column of Table 6 we show the specific dependence, while in the last three columns we collect their determination from the fit to experiment rates and slopes obtained by Ecker et al. [17] for three different scales, i.e. $\mu = m_\eta, m_\rho, 1 \text{ GeV}$.

The first analysis of $\mathcal{O}(p^4)$ weak VMD through vector resonances (no axial-vectors) in $K \rightarrow \pi\pi\pi$ was performed in the hidden symmetry formulation of vector mesons [41], and the FM was then used to evaluate the couplings. However no vector contribution to the direct vertices has been found in this framework.

A complete analysis of vector, axial-vector, scalar and pseudoscalar resonances in the FM has been done in Ref. [17] using the antisymmetric formulation for the spin-1 fields. In that reference no vector nor axial-vector contributions to the direct vertices in $K \rightarrow 2\pi/3\pi$ were found. The scalar and pseudoscalar resonance contributions are shown in the fifth column of Table 6 where k_F is the FM factor.

Our non-vanishing N_i combinations contributing to $K \rightarrow \pi\pi(\pi)$ are shown in Table 7. In the just quoted previous works all the entries in this Table were zero. We get, therefore, new contributions.

We find $N_1^V + N_2^V = 0$ independent of factorization. This same result was also found in Refs. [17, 41] but with the factorization hypothesis. Using the phenomenological values for the strong coupling of vectors and axials in Tables 1 and 2 we obtain a determination of the N_i combinations (see fourth and fifth column in Table 7) and then to counterterms contributing to $K \rightarrow \pi\pi(\pi)$: K_1 , K_2 and K_3 (fourth column in Table 6).

In our framework we find for the first time a direct (not given by L_i 's) vector and axial-vector contribution to $K \rightarrow \pi\pi$ and $K \rightarrow \pi\pi\pi$. In particular it is worth noting

i	K_i	Our model-independent analysis		Ecker et al. analysis [17]			
		V+A (vector formulation)		S+P	Phenomenological result		
		Prediction	Numerical value	(FM)	$\mu = m_\eta$	$\mu = m_\rho$	$\mu = 1 \text{ GeV}$
1	$9G_8F_\pi^2(-N_5^r + 2N_7^r - 2N_8^r - N_9^r)$	$-9\sqrt{2}G_8F_\pi^2f_V\beta_V\eta_V$	$3.65\eta_V$	$-1.70k_F$	0.0	4.5	7.9
2	$3G_8F_\pi^2(N_1^r + N_2^r + 2N_3)$	$-6\sqrt{2}G_8F_\pi^2(\gamma_1 + 2\gamma_2)f_A\eta_A$	$1.05\eta_A$	$1.70k_F$	8.4	7.8	7.3
3	$3G_8F_\pi^2(N_1^r + N_2^r - N_3)$	$12\sqrt{2}G_8F_\pi^2(\gamma_1 - \gamma_2)f_A\eta_A$	$1.83\eta_A$	$1.70k_F$	5.8	5.2	4.7

Table 6: Counterterm combinations K_i relevant for $K \rightarrow \pi\pi/\pi\pi\pi$. The numerical value of our predictions is in units of $[G_8/(9.2 \times 10^{-6} \text{ GeV}^{-2})] \times 10^{-9}$. The numerical values in the Ecker et al. analysis are in units of 10^{-9} . Our analysis gives the vector and axial-vector contributions ($V + A$) in the model-independent framework proposed in the text. In the third column we have put $\gamma_3 = \gamma_4 = 0$ in K_2 and K_3 . The analysis of Ecker et al. [17] gives the contributions of scalar and pseudoscalar ($S + P$) resonances in the FM.

Counterterm	Vectors	Axial-Vectors	Phenomenological result with $\omega_1^R = \sqrt{2} \frac{m_R^2}{F_\pi^2} f_R \eta_R$	
			Vectors	Axial-Vectors
$N_1^r + N_2^r$	-	$2 \frac{F_\pi^2}{m_A^2} (\gamma_1 - 2\gamma_2 - 2\gamma_4) \omega_1^A$	-	$0.007\eta_A$
N_3	-	$-2 \frac{F_\pi^2}{m_A^2} (\gamma_1 + \gamma_3) \omega_1^A$	-	$-0.001\eta_A$
N_9^r	$\frac{F_\pi^2}{m_V^2} \beta_V \omega_1^V$	-	$-0.005\eta_V$	-

Table 7: $\mathcal{O}(p^4)$ counterterms with vector and axial-vector contributions relevant for $K \rightarrow 2\pi/3\pi$.

that K_1 contributes to A_0 of $K \rightarrow \pi\pi$ (see Eq. (30)), which thus gets contributions from vectors in this formulation ⁵.

Before we proceed to establish the conclusions we reach from our results, several cautious remarks concerning the status of the study of the $K \rightarrow \pi\pi(\pi)$ channels have to be made : i) the experimental error in the strong $\mathcal{O}(p^4)$ couplings (L_i) which also contribute to these processes has not been taken into account in the analysis of Refs. [17, 33]; ii) isospin breaking corrections, potentially relevant, have not been studied yet and iii) experiments with better accuracy are needed in order to determine more precisely the $\mathcal{O}(p^4)$ weak counterterm combinations K_i (see a preliminary study in Ref. [43]).

Due to these incertitudes one cannot make a conclusive statement or an accurate quantitative analysis. From our results, however, we think it is conservative to conclude that :

- a) Our contributions improve the phenomenological agreement. If we take a look to our $V + A$ contribution in the fourth column of Table 6, add it to the $S + P$ contribution in the fifth column and compare with the phenomenological values in the last three columns we note that in the three cases K_1 , K_2 and K_3 our contributions aim to improve the original result of Ecker et al. In fact a complete phenomenological agreement, once all contributions are considered, might indicate that $\eta_V > \eta_A$.
- b) As seen in Table 6 there is no splitting between K_2 and K_3 in the FM evaluation of the $S + P$ contributions while the phenomenological values seem to indicate it. Those contributions coming from $V + A$ in our framework show a splitting that, however, goes in the opposite direction to the phenomenological one. We note that K_2 and K_3 get also a contribution from γ_3 and γ_4 in Table 2 that are suppressed by the number of colours expansion and unknown (we have used $\gamma_3 = \gamma_4 = 0$ in Tables 6 and 7). It cannot be excluded that this contribution helps to recover agreement with the phenomenological value of the splitting. Due to the sensitivity of the weak counterterm contributions to $K \rightarrow \pi\pi\pi$ to the γ_i couplings we think that a better control on these couplings is needed.
- c) From our analysis we see that a large η_V and η_A , let us say $\mathcal{O}(1)$ or even larger, fits better the phenomenology of $K \rightarrow \pi\pi(\pi)$. This is not the case, however, for the other decays we have studied, like for instance $K^\pm \rightarrow \pi^\pm \gamma\gamma$ and the anomalous non-leptonic kaon decays [39], where a value close to the Wilson coefficient, i.e. 0.2-0.3, gives better agreement with the phenomenology. Of course at leading order the

⁵We can split the massive spin-1 field propagator $\mathcal{D}_{\mu\nu}(k)$ as [42] $\mathcal{D}_{\mu\nu}(k) = \mathcal{D}_{\mu\nu}^s(k) + \mathcal{D}_{\mu\nu}^t(k)$ where ‘s’ stands for space-like and ‘t’ for time-like polarizations. In this way $k^\mu \mathcal{D}_{\mu\nu}^s(k) = 0$ but $k^\mu \mathcal{D}_{\mu\nu}^t(k) = -k_\nu/m_V^2$, with m_V the mass of the field. We interpret the vector contribution to $K \rightarrow \pi\pi$ as generated by a local term (non-pole) through the time-like part of the vector propagator. This is analogous to the spin-1 W -boson contribution to $\pi \rightarrow \mu(e)\nu$.

vector coupling is unique for all decays, but $\mathcal{L}_4^{|\Delta S|=1}$ is evaluated with couplings at the scale of the resonance. We may interpret these $\eta_V \simeq \eta_A \simeq 1$ in $K \rightarrow \pi\pi(\pi)$, at the phenomenological level, as generated by a non-perturbative enhancement of the ‘running’ couplings η_V and η_A between the scale of the ρ and the m_K , which happens in these particular processes ($K \rightarrow \pi\pi(\pi)$). The same, we know, is happening for G_8 (see discussion in Section 2.3).

As a conclusion we find that, in contradistinction to what happens in the antisymmetric formulation [17] or the hidden symmetry [41], VMD goes in the direction addressed by the phenomenology if the conventional vector formulation is employed. To disentangle which one is the right approach it would be crucial an accurate phenomenological determination of the K_i terms and, in particular, of K_1 .

4.2 Non-anomalous radiative non-leptonic kaon decays

Our formulation gives new insights in several of these processes too.

$K \rightarrow \pi\gamma^*$

$K^\pm \rightarrow \pi^\pm\gamma^*$ and $K_S \rightarrow \pi^0\gamma^*$ get their leading contribution in χ PT at $\mathcal{O}(p^4)$ with loops and the following counterterm combinations [44] :

$$\begin{aligned}\omega_+ &= \frac{64\pi^2}{3} [N_{14}^r - N_{15}^r + 3L_9^r] + \frac{1}{3} \ln\left(\frac{\mu^2}{m_K m_\pi}\right), \\ \omega_S &= \frac{32\pi^2}{3} [2N_{14}^r + N_{15}^r] + \frac{1}{3} \ln\left(\frac{\mu^2}{m_K^2}\right),\end{aligned}\tag{31}$$

where μ is the renormalization scale and cancels the scale dependence of the counterterm combinations. Since we use vector meson saturation $\mu \simeq m_\rho$.

No useful experimental information exists on $K_S \rightarrow \pi^0\gamma^*$, but DAΦNE [1] will either measure the branching ratio or will put an interesting bound, which it will turn in a non-trivial constrain on ω_S . ω_+ has been measured in the two possible lepton final states: muon and electron. Here ω_+ can be extracted from the rate and the spectrum. In $K^\pm \rightarrow \pi^\pm e^+ e^-$ the value extracted for ω_+ from a combined fit of the rate and spectrum is : $0.89_{-0.14}^{+0.24}$ [45]. If only the rate is considered the result is $\omega_+ = 1.20 \pm 0.04$ [45, 46]. From the rate of $K^\pm \rightarrow \pi^\pm \mu^+ \mu^-$ the value $\omega_+ = 1.07 \pm 0.07$ is obtained [46]. The measured ratio $\Gamma(K^\pm \rightarrow \pi^\pm \mu^+ \mu^-)/\Gamma(K^\pm \rightarrow \pi^\pm e^+ e^-)$ is over 2 σ ’s away from the $\mathcal{O}(p^4)$ χ PT prediction [46] for a wide range of values of ω_+ , which includes the ones given above. In our formulation the relations for these counterterms are independent of factorization (see Eqs. (26,27) and Table 4). Also we have a new axial–vector contribution (see Table 4) proportional to α_A . With the values in Tables 1 and 2 we obtain

$$\omega_+ = 4.4 - 4.2\eta_V + 0.8\eta_A \quad , \quad \omega_S = 0.3 + 8.4\eta_V .\tag{32}$$

η_V	$Br(K_S \rightarrow \pi^0 e^+ e^-)$	$Br(K_S \rightarrow \pi^0 \mu^+ \mu^-)$
0.3	1.8×10^{-8}	3.9×10^{-9}
0.5	5.1×10^{-8}	1.1×10^{-8}
1.0	2.1×10^{-7}	4.4×10^{-8}

Table 8: Branching ratios of $K_S \rightarrow \pi^0 \ell^+ \ell^-$ ($\ell = e, \mu$) for different values of η_V using ω_S in Eq. (32).

Thus we expect some cancellation for ω_+ among weak and strong vector contributions. A value $\eta_V \simeq 1$ could still accommodate the phenomenology of this channel and the one of $K \rightarrow \pi\pi(\pi)$, as commented in the previous Subsection. However such solution might be in contradiction with the value of η_V , which can be deduced, as we shall see, from $K^\pm \rightarrow \pi^\pm \gamma\gamma$ and from the anomalous non-leptonic kaon decays (see also Ref. [39]).

We think it is premature a final conclusion on the value of ω_+ due to the possible disagreement among the $\mathcal{O}(p^4)$ χ PT spectrum and experiments. As it happens in $K_L \rightarrow \pi^+ \pi^- \gamma$ $\mathcal{O}(p^6)$ contributions could be important [39] (see the discussion on $\mathcal{O}(p^6)$ corrections to G_8 in Ref. [47]).

From the expression of ω_S in Eq. (32) we notice that $K_S \rightarrow \pi^0 \ell^+ \ell^-$ is very sensitive to the value of η_V . In Table 8 we show the predictions for the branching ratio for three representative values of η_V . DAΦNE should be crucial for a better phenomenological understanding of this process since it should be able to measure $Br(K_S \rightarrow \pi^0 e^+ e^-) > 5 \times 10^{-10}$ and $Br(K_S \rightarrow \pi^0 \mu^+ \mu^-) > 10^{-10}$ [1].

In Ref. [17] a different approach has been proposed. Using procedure \mathcal{A} of Section 2.3 for the counterterm combinations in Eq. (32) they obtain ⁶ [1] :

$$\begin{aligned}
\omega_+ &= 5.3 - 6.9k_F - \frac{(16\pi)^2}{3} k_F H_1^r(m_\rho) , \\
\omega_S &= 0.3 + 2.3k_F - \frac{(16\pi)^2}{3} k_F H_1^r(m_\rho) , \\
\omega_S &= \omega_+ + 4.6(2k_F - 1) - 0.43 .
\end{aligned} \tag{33}$$

If we use VMD for H_1 the same result than ours in Eq. (32) would be obtained (with $\eta_V = \eta_A = k_F$ and neglecting the new axial contribution proportional to α_A) ⁷. However in Ref. [17] the choice $H_1^r(m_\rho) = 0$ has been advocated. This is a clear departure from VMD and must be considered a local contribution. Of course, if the suggestion $H_1 = 0$

⁶We note that there is a numerical difference between the model independent contribution to ω_+ in Eqs. (32,33) and also the factor multiplying the parentheses in Eq. (34) (below) and the analogous one in Eq. (33). This is due to the fact that we use different values for the f_V coupling.

⁷VMD gives $H_1^r(m_\rho) = -(f_V^2 + f_A^2)/8$. This result is consistent with the evaluation of H_1 in the ENJL model [29].

is supported by the phenomenology, we can implement it in our scheme by adding a suitable local contribution. Notice that the last relation in Eq. (33) is independent of the H_1 coupling and arises in procedure \mathcal{A} as a consequence of factorization. In our formulation this relation translates into

$$\omega_S = \omega_+ + 6.3(2\eta_V - 1 - 0.12\eta_A) - 0.43, \quad (34)$$

but now it is independent of factorization, hence we conclude that theirs too. Also chiral symmetry tells us that a possible additional local term, if any, generated by H_1 (to be included into ω_+ and ω_S) cancels in this relation.

Our insight in these channels is to have clarified that the previous results (that coincide with ours if we assume VMD and neglect axial–vector contributions) are model–independent and do not rely on factorization (see for instance Eq. (34)).

$\mathbf{K}^\pm \rightarrow \pi^\pm \gamma \gamma$

Due to the recent BNL measurement [48] this channel is now particularly interesting. It starts at $\mathcal{O}(p^4)$ in χ PT with a finite loop contribution and the scale independent counterterm combination [49]

$$\hat{c} = \frac{128\pi^2}{3} [3(L_9 + L_{10}) + N_{14} - N_{15} - 2N_{18}] , \quad (35)$$

where L_9 and L_{10} are couplings in the $\mathcal{O}(p^4)$ χ PT strong lagrangian \mathcal{L}_4 [9]. There is no complete study at $\mathcal{O}(p^6)$ but the unitarity corrections from $K \rightarrow \pi\pi\pi$ have been computed [50] and enhance the $\mathcal{O}(p^4)$ rate by 30 – 40%. Vector exchange at this order has been proven to be negligible [28, 50]. The BNL measurement in fact has a better fit with these $\mathcal{O}(p^6)$ contributions [48]. At $\mathcal{O}(p^4)$ the vector contribution cancels in the strong (L_i) and in the weak sector (N_i) (see Table 4). In the formulation used in this paper this result is independent of factorization. Using the values in Tables 1 and 2 and the expressions in Table 4 we get

$$\hat{c} = 2.15 - 4.2\eta_A . \quad (36)$$

Thus \hat{c} is sensitive to the weak coupling of axials and hence α_A (see Table 4) might correct the result in Ref. [17]. We observe that the recent experimental figure $\hat{c} = 1.8 \pm 0.6$ [48] (this is the value obtained when unitarity corrections have been included, as supported by a better χ^2) prefers a small but non–vanishing value $\eta_A \simeq 0.2 - 0.3$. Thus the perturbative evaluation seems consistent with data. Even a slight improvement in the actual experimental situation would fix unambiguously η_A .

$\mathbf{K} \rightarrow \pi\pi(\pi)\gamma$

In the amplitudes of these processes one generally separates the bremsstrahlung contribu-

tion, which can be related to the non-radiative amplitude by the Low Theorem [19, 51], and the direct emission (structure dependent) component to $K \rightarrow \pi\pi(\pi)\gamma$ [1, 5, 6]. From the spectrum in the photon energy the two contributions can be distinguished.

The direct emission amplitudes, according to their transformation under parity, are divided in electric and magnetic contributions [52] and get their first non-vanishing contributions at $\mathcal{O}(p^4)$ in χ PT. The magnetic terms come from odd-intrinsic parity terms and will be commented in the next Subsection.

Only one $\mathcal{O}(p^4)$ weak counterterm combination appears for $K \rightarrow \pi\pi\gamma$ decays: $N_{14} - N_{15} - N_{16} - N_{17}$ (see Table 9) and it is scale independent [18, 34, 35, 36]. Thus the chiral loop contribution is finite. As we can see from Tables 4 and 9, we have an almost complete cancellation of vector contributions, independently of factorization. Our axial-vector contributions (and consequently the bulk of the total contribution) are about 50% bigger than the result of Ref. [17], due to our different results for N_{16}^A and N_{17}^A .

The electric amplitudes for $K_L(K^\pm) \rightarrow \pi\pi\pi\gamma$ also get the previous counterterm combination [19]. Meanwhile for $K_S \rightarrow \pi^+\pi^-\pi^0\gamma$ the relevant $\mathcal{O}(p^4)$ scale dependent counterterm combination is $7(N_{14}^r - N_{16}^r) + 5(N_{15}^r + N_{17}^r) \simeq 0.39\eta_V + 0.01\eta_A$ where we have quoted our spin-1 exchange contributions. Our value is larger (30%) than the one obtained with factorization in the antisymmetric formulation [17, 19] due to our different result for N_{16}^A and N_{17}^A .

In Table 9, the full list of $\mathcal{O}(p^4)$ VMD contributions is reported. Our results will differ particularly from Ref. [17], when axials are important and N_{16}^A and N_{17}^A are involved. This happens, for instance, in the channel $K_L \rightarrow \pi^+\pi^-\gamma^*$, which is relevant now due the preliminary data from Fermilab [53], which for the first time measures this decay. This $\mathcal{O}(p^4)$ electric contribution is in competition with the bremsstrahlung CP violating and the magnetic amplitudes.

For a thorough review on these processes see the Refs. [1, 5, 6, 19, 34].

4.3 Anomalous radiative non-leptonic kaon decays

In our spin-1 formulation the weak couplings associated to anomalous transitions (N_{28}, \dots, N_{31}) are particularly interesting. Contrarily to the antisymmetric formulation [17], we do get vector and axial-vector contributions to these counterterms (see Table 5). This should be interpreted in the same terms as the failure of the antisymmetric formulation of the vectors to describe the strong lagrangian $V \rightarrow P\gamma$ (see discussion in section 2.1) at the proper chiral order.

Due to previous studies in factorization [37] it has become customary to rewrite these couplings as

$$\begin{aligned} a_1 &= 8\pi^2 N_{28} & , & & a_2 &= 32\pi^2 N_{29} , \\ a_3 &= \frac{16}{3}\pi^2 N_{30} & , & & a_4 &= 16\pi^2 N_{31} . \end{aligned} \tag{37}$$

Counterterm combination	Processes	Phenomenological result with $\omega_1^R = \sqrt{2} \frac{m_B^2}{F_\pi^2} f_R \eta_R$, $\omega_2^R = -\omega_1^R$
$N_{14}^r - N_{15}^r$	$K^+ \rightarrow \pi^+ \gamma^*$ $K^+ \rightarrow \pi^+ \pi^0 \gamma^*$	$-0.020 \eta_V + 0.004 \eta_A$
$2N_{14}^r + N_{15}^r$	$K_S \rightarrow \pi^0 \gamma^*$ $K_L \rightarrow \pi^0 \pi^0 \gamma^*$	$0.08 \eta_V$
$N_{14} - N_{15} - 2N_{18}$	$K^+ \rightarrow \pi^+ \gamma \gamma$ $K^+ \rightarrow \pi^+ \pi^0 \gamma \gamma$ $K_S \rightarrow \pi^+ \pi^- \gamma \gamma$	$-0.01 \eta_A$
$N_{14} - N_{15} - N_{16} - N_{17}$	$K^+ \rightarrow \pi^+ \pi^0 \gamma$ $K_S \rightarrow \pi^+ \pi^- \gamma$ $K^+ \rightarrow \pi^+ \pi^+ \pi^- \gamma$ $K^+ \rightarrow \pi^+ \pi^0 \pi^0 \gamma$ $K_L \rightarrow \pi^+ \pi^- \pi^0 \gamma$	$0.002 \eta_V - 0.010 \eta_A$
$7(N_{14}^r - N_{16}^r) + 5(N_{15}^r + N_{17})$	$K_S \rightarrow \pi^+ \pi^- \pi^0 \gamma$	$0.39 \eta_V + 0.01 \eta_A$
$N_{14}^r - N_{15}^r - 3(N_{16}^r - N_{17})$	$K_L \rightarrow \pi^+ \pi^- \gamma^*$	$-0.004 \eta_V + 0.018 \eta_A$
$N_{14}^r - N_{15}^r - 3(N_{16}^r + N_{17})$	$K_S \rightarrow \pi^+ \pi^- \gamma^*$	$0.05 \eta_V - 0.04 \eta_A$
$N_{14}^r + 2N_{15}^r - 3(N_{16}^r - N_{17})$	$K^+ \rightarrow \pi^+ \pi^0 \gamma^*$	$0.12 \eta_V + 0.01 \eta_A$
$N_{29} + N_{31}$	$K_L \rightarrow \pi^+ \pi^- \gamma$ $K^+ \rightarrow \pi^+ \pi^+ \pi^- \gamma$ $K_S \rightarrow \pi^+ \pi^- \pi^0 \gamma$	$0.005 \eta_V + 0.003 \eta_A$
$3N_{29} - N_{30}$	$K^+ \rightarrow \pi^+ \pi^0 \gamma$ $K^+ \rightarrow \pi^+ \pi^0 \pi^0 \gamma$	$-0.005 \eta_V - 0.003 \eta_A$
$5N_{29} - N_{30} + 2N_{31}$	$K_S \rightarrow \pi^+ \pi^- \pi^0 \gamma$	$0.005 \eta_V + 0.003 \eta_A$
$6N_{28} + 3N_{29} - 5N_{30}$	$K_L \rightarrow \pi^+ \pi^- \pi^0 \gamma$	$-0.004 \eta_V - 0.003 \eta_A$

Table 9: Counterterm combinations appearing in radiative non-leptonic kaon decays and the vector and axial-vector contribution to them. The assumption of factorization ($\omega_2^R = -\omega_1^R$) has been used only in the terms involving N_{28}, \dots, N_{31} . For the numerical results we use the experimental value of the couplings, when available, and the ENJL predictions in Tables 1 and 2. However in the last counterterm combination we notice that the coefficient of the vector contribution is very sensitive to the value of the θ_V coupling chosen, i.e. if we had chosen $\theta_V = 2h_V$ (HS prediction) the coefficient would be $+0.037$ instead of -0.004 . This is the only entry where the two model predictions for the strong couplings in Tables 1 and 2 give a substantially different result.

The a_i are positive parameters of $\mathcal{O}(1)$. As pointed out in relation with the Factorization Model in Subsection 2.3 these couplings get a factorizable contribution due the Wess–Zumino–Witten anomaly action [36]. If factorization works this contribution would be the same for all a_i ’s in Eq. (37) that we call $a_i = \eta_{FM}$, $i = 1, 2, 3, 4$. To these we should add the vector and axial–vector contributions that we have found in our work.

The octet operators W_{28}, \dots, W_{31} are relevant for several processes: $K \rightarrow \pi\pi\gamma$ [38, 39], $K \rightarrow \pi\pi\pi\gamma$ [19, 38], etc. For the complete analysis of $K_L \rightarrow \pi^+\pi^-\gamma$ at $\mathcal{O}(p^6)$ we refer to our work [39]. Regarding the $K^\pm \rightarrow \pi^\pm\pi^0\gamma$ process, if one neglects higher order contributions, one can write the $\mathcal{O}(p^4)$ amplitude as [37]

$$A_4 \equiv -(2 - 3a_2 + 6a_3) \simeq -(2 + 3\eta_{FM} + 1.7\eta_V + 0.9\eta_A). \quad (38)$$

The measured branching ratio [27] implies (neglecting $\mathcal{O}(p^6)$ corrections) $A_4 = -4.5 \pm 0.5$ [38]. Thus our new contributions go again in the right direction.

We could conclude that the phenomenology of the aforementioned decays prefers a small value of the factorization factors (η_{FM} , η_V , η_A), i.e., inside the errors, they can be identified with the Wilson coefficient in Eq. (23).

We refer to Table 9 for a complementary list of the counterterm contributions relevant for other radiative non–leptonic kaon decays and our predictions for them.

5 Conclusions

We have given a complete description of the spin-1 meson exchange contributions to the $\mathcal{O}(p^4)$ weak chiral lagrangian in the conventional vector formulation. A detailed comparison with the results obtained in the antisymmetric formulation [17] has also been developed.

In the comparison we find the following observations and new results :

- i) In our formulation the leading weak lagrangian for the vectors (see Eq. (24)) is rather constrained, while the strong sector (see Eqs. (11,13)) is very rich. This is at odds with the antisymmetric formulation, where the opposite is true: constrained strong lagrangian and rich weak lagrangian. This is due to the relatively different chiral power at which both sectors (strong and weak) get their leading contribution. Since we know better (from a phenomenological point of view) the strong and electromagnetic decays of vector mesons, we think that our procedure is in better shape to give more information with less assumptions. In this work and when necessary we have relied on models (typically ENJL [26]) to get the strong couplings, but these are more under control than the weak couplings of vectors and in any case experiments should be able to determine the full strong sector in the near future. Due to our very restrictive weak lagrangian for vectors and axial–vectors, all the

results obtained in factorization in the antisymmetric formulation are obtained by us but without the use of factorization (for N_{16}^A and N_{17}^A see Section 3). We think that this is due to the fact that chiral symmetry and Lorentz invariance imposes to the $\mathcal{O}(p)$ weak lagrangian linear in the resonance fields $\mathcal{L}_R^{\mathcal{O}(p)}$ in Eq. (24) the factorizable structure not as a hypothesis of work but as a consequence of the symmetry and therefore model-independent. We have got new contributions and shown their phenomenological relevance. This gives a more solid ground to these results and complements the analysis of Ref. [17].

- ii) We emphasize that our results for $K \rightarrow \pi\pi(\pi)$ and non-anomalous radiative decays are completely model-independent. The hypothesis of factorization only enters (and with a very minor rôle) in the processes involving N_{28}, \dots, N_{31} , and therefore anomalous radiative kaon decays. Due to our very simple, constrained and also general procedure (factorization is used just to relate N_{30}^V to the other N_i^V and N_{31}^A to the other N_i^A), we parameterize all the spin-1 resonance contributions to $\mathcal{O}(p^4)$ weak lagrangian in terms of only two physical weak couplings of vectors and axials: η_V and η_A (see Table 9).
- iii) We obtain non-vanishing vector and axial-vector exchange contributions in odd-intrinsic parity terms. These appear only at higher orders in the antisymmetric formulation. This problem was already realized in the strong sector for the $VP\gamma$ vertex, contributing at $\mathcal{O}(p^6)$ to $\gamma\gamma \rightarrow \pi^0\pi^0$ in the vector formulation and $\mathcal{O}(p^8)$ in the antisymmetric formulation, in contradiction with QCD properties [15].
- iv) Regarding the $\mathcal{O}(p^4)$ axial-vector contributions to N_{16} and N_{17} we differ from the ones in Ref. [17] and this is phenomenologically relevant.
- v) We get new contributions to the N_i , due to our richer strong lagrangian: all the terms proportional to $\alpha_{V,A}$, β_V , and γ_i are new. The ones to $K \rightarrow \pi\pi(\pi)$ seem to us of immediate phenomenological impact and thus here we have made a more detailed study of the phenomenological implications.
- vi) We have also commented that in an explicit analysis of $K_L \rightarrow \pi^+\pi^-\gamma$, where $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ VMD contributions were simultaneously present, our approach and the vector realization give a theoretical consistent and also phenomenologically successful picture. Actually it is shown that our treatment of weak $\mathcal{O}(p^4)$ VMD is compulsory in order to have the correct description of weak $\mathcal{O}(p^6)$ VMD [39].

Experiments should decide which of the formulations is better. Wherever this is not possible we suggest to choose the one which is able to describe more physics at the lowest

order such to have a better convergence.

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Appendix A : The weak ω_i^R couplings in the Factorization Model in the Vector couplings

The ω_i^R weak couplings defined in Eq. (24) for $R = V, A$ can be evaluated in the FMV we have proposed in Ref. [28]. The bosonization of the Q_- operator in Eq. (20) can be carried out in the FMV from the strong action S of a chiral gauge theory. If we split the strong action and the left-handed current into two pieces : $S = S_1 + S_2$ and $\mathcal{J}_\mu = \mathcal{J}_\mu^1 + \mathcal{J}_\mu^2$, respectively, the Q_- operator is represented, in the factorization approach, by

$$Q_- \leftrightarrow 4 \left[\langle \lambda \{ \mathcal{J}_\mu^1, \mathcal{J}_\mu^2 \} \rangle - \langle \lambda \mathcal{J}_\mu^1 \rangle \langle \mathcal{J}_\mu^2 \rangle - \langle \lambda \mathcal{J}_\mu^2 \rangle \langle \mathcal{J}_\mu^1 \rangle \right] , \quad (\text{A.1})$$

with $\lambda \equiv (\lambda_6 - i\lambda_7)/2$ and, for generality, the currents have been supposed to have non-zero trace.

In order to apply this procedure to construct the factorizable contribution to the weak $\mathcal{O}(p)$ lagrangian in Eq. (24) we have to identify in the full strong action the pieces that can contribute at this chiral order. We define, correspondingly,

$$S = S_{R\gamma} + S_2^\chi , \quad (\text{A.2})$$

where the actions correspond to the lagrangian densities proportional to f_V in \mathcal{L}_V (Eq. (11)) or to f_A in \mathcal{L}_A (Eq. (13)) ($S_{R\gamma}$) and \mathcal{L}_2 in Eq. (4) (S_2^χ).

Evaluating the left-handed currents and keeping only the terms of interest we get

$$\begin{aligned} \frac{\delta S_{R\gamma}}{\delta \ell^\mu} &= -\frac{f_R}{\sqrt{2}} m_R^2 u^\dagger R_\mu u , \\ \frac{\delta S_2^\chi}{\delta \ell^\mu} &= -\frac{F_\pi^2}{2} u^\dagger u_\mu u . \end{aligned} \quad (\text{A.3})$$

Then the effective action in the factorization approach is

$$\begin{aligned} \mathcal{L}_R^{fact} &= 4 G_8 \eta_R \left[\langle \lambda \left\{ \frac{\delta S_{R\gamma}}{\delta \ell^\mu}, \frac{\delta S_2^\chi}{\delta \ell_\mu} \right\} \rangle - \langle \lambda \frac{\delta S_{R\gamma}}{\delta \ell^\mu} \rangle \langle \frac{\delta S_2^\chi}{\delta \ell_\mu} \rangle \right. \\ &\quad \left. - \langle \lambda \frac{\delta S_2^\chi}{\delta \ell^\mu} \rangle \langle \frac{\delta S_{R\gamma}}{\delta \ell_\mu} \rangle \right] + h.c. , \end{aligned} \quad (\text{A.4})$$

and identifying with $\mathcal{L}_R^{\mathcal{O}(p)}$ in Eq. (24) we read

$$\omega_1^R = \sqrt{2} \frac{m_R^2}{F_\pi^2} f_R \eta_R , \quad \omega_2^R = -\omega_1^R , \quad (\text{A.5})$$

for $R = V$ and A .

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